

Emergence of symmetry from random n -body interactions

Alexander Volya

Department of Physics, Florida State University, Tallahassee, FL 32306-4350, USA

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An ensemble with random n -body interactions is investigated in the presence of symmetries. A striking emergence of regularities in spectra, ground state spins and isospins is discovered in both odd and even-particle systems. Various types of correlations from pairing to spectral sequences and correlations across different masses are explored. A search for interpretation is presented.

Recent progress in *ab-initio* treatment of light nuclei show unambiguously the essential role played by the three and four-body forces [1, 2]. With the advancement of the mesoscopic physics and with the ability to manipulate interactions the question how n -body interactions play out in the many-body physics becomes increasingly important.

It is known since the Wigner-Dyson random matrix theory (RMT) [3], see also [4, 5], that complex configuration mixing driven by many-body forces have generic statistical features which also depend on the nature of interaction [6]. Symmetries have a robust effect, which is not fully understood. Recently, a remarkable geometrical ordering have been numerically found in the Two-Body Random Ensembles (2-BRE)[7]. The main question is centered around the disproportionately large probability for the ground state (g.s.) spin J to be zero, $J_0 = 0$, in the even-particle system. Manifestations of the symmetry were seen in features of the mean-field [8, 9], and in transition probabilities indicating vibrational and rotational low-lying spectra [10]. Starting from the pioneering paper [7] over a hundred works were published by different groups striving to understand the emergence of symmetries, their role and origin in the 2-BRE. The summary of these efforts may be found in reviews [10, 11, 12]. The success is mixed, we understand that pairing [13] and time-reversal [14] are not the primary causes. The boson approximation of fermion pairs and chaos arising from complex geometrical couplings, the *geometric chaoticity*, provide some qualitative understanding of the trend; more quantitative explanations have been suggested in Refs. [15, 16] via numerical observations of geometrical correlations and diagonalization of the individual interaction terms. Nevertheless, the simple question of symmetry and chaos asks for a simple answer which is still missing. Driven by the quest for understanding of interplay between symmetry and many-body complexity and the general interest and importance of many-body interactions in modern physics in this work we address the n -body Random Ensembles (n -BRE) with symmetries. The evolution of spectrum and level spacing for the Gaussian n -BRE without symmetries is discussed in Ref. [6].

The n -body rotationally invariant interaction Hamil-

tonian is defined as

$$H^{(n)} = \sum_{\alpha\beta} \sum_L V_L^{(n)}(\alpha\beta) \sum_{M=-L}^L T_{LM}^{(n)\dagger}(\alpha) T_{LM}^{(n)}(\beta),$$

where operators $T_{LM}^{(n)\dagger}(\alpha)$ are n -body creation operators coupled to a total angular momentum L and magnetic projection M , $T_{LM}^{(n)\dagger}(\alpha) = \sum_{12\dots n} C_{12\dots n}^{LM}(\alpha) a_1^\dagger a_2^\dagger \dots a_n^\dagger$, here 1 is the single-particle index. For $n = 2$ the coefficients C_{12}^{LM} are proportional to the Clebsch-Gordan coefficients and index α is uniquely defined by single-particle levels involved. For $n > 2$ the index α must include additional information about the coupling scheme. Here we define the basis set of normalized n -body operators $T_{LM}^{(n)}(\alpha)$ from a full set of orthogonal n -body eigenstates $T_{LM}^{(n)\dagger}(\alpha)|0\rangle$ of an arbitrarily chosen reference two-body Hamiltonian $H_0^{(2)}$, solved numerically. The exact form of this Hamiltonian is irrelevant as long as it preserves rotational and other symmetries of the problem.

The n -BRE is defined as a set of n -body interaction Hamiltonians that for an n -particle system leads to a Gaussian Orthogonal Ensemble (GOE) within every symmetry class. The interaction strengths $V_L^{(n)}(\alpha, \beta)$ are selected at random with normal distribution centered at zero $\langle V_L^{(n)}(\alpha, \beta) \rangle = 0$ and unit diagonal variance $\langle V_L^{(n)}(\alpha, \beta) V_L^{(n)}(\alpha', \beta') \rangle = \delta_{LL'} \delta_{\alpha\alpha'} \delta_{\beta\beta'} (1 + \delta_{\alpha\beta})/2$. The time reversal symmetry sets $V_L^{(n)}(\alpha, \beta) = V_L^{(n)}(\beta, \alpha)$, thus we assume ordered labels $\alpha \geq \beta$. The ensemble is $H_0^{(2)}$ independent. The variance defines the energy unit. Our ensemble extends the n -body Embedded GOE [17, 18] by incorporating symmetries. Inclusion of the isospin symmetry is straightforward.

Through the text we use P [event] to denote chances of observing a certain event in the n -BRE. Most commonly we discuss $J(N)_0$ which is an event where g.s. of an N -particle system has spin J ; the subscript reflects the order in excitation energy, 0-for g.s. 1-first excited state, and etc. Where obvious we omit the explicit reference to the particle number N .

We start with a single j -level model of $2j + 1$ degeneracy with N identical fermions. In Fig. 1 some examples are shown. In the special case of $n = 1$, the mean-field only, all many-body states are degenerate. Here by definition we assume an equiprobable ordering of the states in the spectrum. Thus, the probability to observe

a g.s. with a spin J , $P[J_0]$ is by definition proportional to the number of many-body states with this spin in the model space $d(J)$. Following the semi-classical consideration of the random-walk-type vector couplings, see also [19] $d(J) \sim (2J+1) \exp[-3J(J+1)/2Nj(j+1)]$ which disfavors both $J = 0$ and the maximum possible spin $J_{max} = N(2j+1-N)/2$. The equiprobable ordering is in drastic contrast to the 2-BRE where g.s. is most likely to be $J_0 = 0$ and the probability for $J_0 = J_{max}$ is large, Fig. 1. The preponderance of $P[0_0]$ becomes stronger for the n -BRE with higher n , and at the same time the chances of J_{max} as g.s. quickly diminish. For $n = 4$ and 5, apart from the dominant $J_0 = 0$, the states with spins $J_0 = 2, 4, 6, 8$ have small, few percent-level, chances to appear as g.s.

The middle panel in Fig. 1 corresponds to the $N = 7$ odd-particle system. The lowest bar in the stacked histogram shows the $P[j_0]$ (where $j = 19/2$) that can be interpreted as a single particle coupled to the $J = 0$ $N = 6$ core. While for the 2-BRE $P[19/2_0] = 12.4(4)\%$ (note that $P[13/2_0] = 14.5(4)\%$), the $P[19/2_0]$ is bigger for $n = 3, 4$ and 5. The maximum spin probability $P[(91/2)_0]$ is enhanced for the 2-BRE but similarly to the even-particle system declines for larger n . Here, and on some occasions below we include statistical errors due to a limited number of random realizations tested.

An important case of $n = N$ is not shown in Fig. 1 because of its drastic contrast to the $n < N$ situations. Here, within each symmetry class the Hamiltonian matrix is represented by the GOE for which with increasing dimensionality $d(J)$ the distribution of eigenvalues quickly converges to the Wigner semicircle with the radius $\sqrt{2d(J)}$. A detailed quantitative analysis can be done using the RMT but it is clear that the probability $P[J_0]$ strongly favors those spins J with the highest dimensionalities $d(J)$. For example, for $N = 8$ $j = 19/2$ the highest dimensions are 179, 178, 173, and 169 for spins $J = 12, 14, 16$, and 10; the corresponding probabilities $P[J_0]$ in the 8-BRE are 41, 35, 13 and 5%, respectively. For lower $n = N - 1$ the change in the spin statistics of g.s. occurs abruptly.

Although pairing was ruled out as an explanation for the 2-BRE given its dominance in realistic systems it is worth revisiting this question for the n -BRE. In Fig. 2 the evolution of seniority s (the number of unpaired particles) as a function of n is shown. The seniority s is evaluated using expectation value in the state of interest $|N, \alpha\rangle$ of the $L = 0$ pair operator, which is unique in a single- j $\langle N, \alpha | T_{00}^{(2)\dagger} T_{00}^{(2)} | N, \alpha \rangle = (N - s)(2j + 3 - N - s) / \{2(2j + 1)\}$.

Surprisingly, the paired state is favored by the three-body forces. Significant lowering of the seniority for $n = 3$ is a robust result in all cases considered Fig. 2. Given that the $P[0_0]$ goes up for higher n while s is increasing for $n > 3$ pairing still does not explain the preponderance of the zero spin g.s. The structure of the two-body forces on a single- j level is known to facilitate seniority conservation, only about a third of the $j - 1/2$ in-

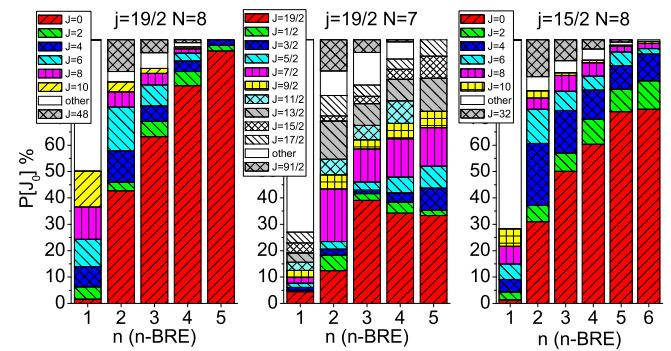


Figure 1: (Color online) The stacked-bar histogram showing the distribution of ground states with a given angular momentum J for a single j -system, with j and N as marked. The histograms are shown as a function of n for different n -BRE. For the $n = 1$ case we assign probability proportionally to the number of states with that spin. The stacking order is marked, and corresponds to the increasing J starting from J_{min} , with the exception of the odd- N where $J_{min} \equiv j$. The spins J with $P[J_0] < 2\%$ are not separately identified, their cumulative probability is shown with the white bar, labeled “other”.

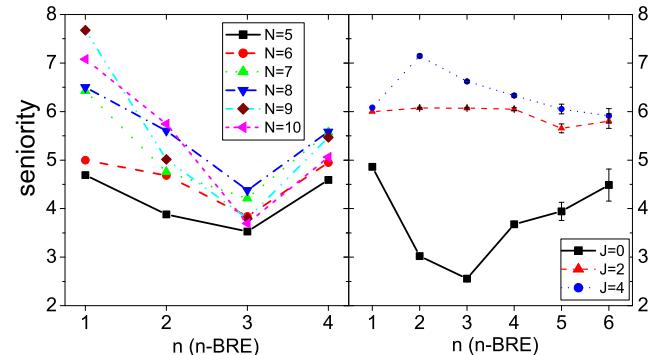


Figure 2: The ensemble averaged seniority s is shown as a function of n for the n -BRE. Left panel corresponds to $j = 19/2$ with connected by line points for different particle numbers. Only $(J_{min})_0$ ground state realizations are selected, namely $J_0 = 0$ for N even and $J_0 = 19/2$ for odd. On the right the half-occupied system $N = 8$ $j = 15/2$ is examined, which shows the average seniority in realizations with g.s $J_0 = 0, 2$ and 4. For the $n = 1$ case, (equiprobable ground state distribution) the average seniority of states with a given spin is quoted.

dependent linear combinations of interaction parameters $V_L^{(2)}$ mix seniorities, seniority is a good quantum number for any interaction on $j < 9/2$ [20]. There is lowering of seniority against the average ($n = 1$) for $n = 2$ and perhaps explanation for $n = 3$ pairing enhancement lies in a similar seniority conserving structure of the three-body interactions. This question is a subject for future work.

The particle number N and the Casimir operators for the symmetry groups such as J^2 and T^2 are conserved quantities, and in each random realization of the Hamil-

tonian are expected to describe its coherent part. In the 2-BRE the particle-hole transformation can be used to obtain a coherent J^2 -dependent component of the Hamiltonian $\langle H^{(2)} \rangle_J = \tilde{V}_1 J^2$. The moment of inertia coefficient \tilde{V}_1 is given via angular momentum recoupling coefficients, for a single- j it is

$$\tilde{V}_1 = \sum_L \frac{3(2L+1)}{j(j+1)(2j+1)} \left\{ \begin{array}{ccc} j & j & 1 \\ j & j & L \end{array} \right\} V_L^{(2)}. \quad (1)$$

The 2-BRE g.s. spin systematics follows from here; see also geometric chaoticity arguments [13]. The \tilde{V}_1 , being a sum of random numbers with normal distribution centered at zero, is itself a normally distributed random variable with equal chances of being positive and negative. Thus the $P[0_0] = P[(J_{max})_0] = 1/2$. This prediction qualitatively describes the observations, although it is distorted by other incoherent interaction terms.

The emergence of the moment of inertia in each realization of interaction is further supported by the mass number independence of the 2-BRE results. Indeed, the moment of inertia (1) is particle-number independent, thus for a given realization of the two-body Hamiltonian if $\tilde{V}_1 > 0$ the g.s. spin is zero $J(N)_0 = 0$ for any even N (j for odd); similarly for $\tilde{V}_1 < 0$ the g.s. is $J_{max}(N)$. To quantify the correlation between the statistics for different N we consider the joint probability that all systems from $N = 5$ to half-occupied, simultaneously have the maximum spin $P[J_{max}(5)_0, J_{max}(6)_0, \dots]$. The same can be done for the minimum spin J_{min} which we define to be zero for an even N and s.p. j for odd. Because of the particle-hole symmetry the eigenvalues of the $2j + 1 - N$ system apart from a monopole constant shift in energy are identical to the N -particle system. In an uncorrelated case the joint probability is a product of the independent probabilities $P[J(N_1)_0, J(N_2)_0, \dots] = \prod_i P[J(N_i)_0]$ which is generally small. On the other hand, if strong correlations exist the joint probability is of the order of individual $P[J_0]$. In our example $j = 19/2$ with $N = 5, 6, \dots, 10$ $P[J_{max}(5)_0 \dots J_{max}(10)_0] = 6.6\%$ in the 2-BRE, while uncorrelated product is four orders of magnitude smaller $P[J_{max}(5)_0] P[J_{max}(6)_0] \dots P[J_{max}(10)_0] = 2.1 \cdot 10^{-4}\%$. In Tab. I we show the total weighted correlation which for a general set of events $J, J', J'', J''' \dots$ is defined as

$$\mathcal{C}[J, J', J'', \dots] = \frac{\log(P[J]P[J']P[J'']\dots)}{\log(P[J, J', J'', \dots])} - 1. \quad (2)$$

If ground states in k systems with different masses are not correlated then $\mathcal{C} = 0$. While in the case of full correlation the ratio of logarithms is approximately equal to k and $\mathcal{C} = k - 1$, meaning that the g.s. spin in one system is sufficient to predict g.s. spins in all other $k - 1$ different mass-systems.

The situation with n -body forces is more complex as coherent higher order terms appear. For the 3-BRE $\langle H^{(2)} \rangle_J = (\tilde{V}'N + \tilde{V}_1)J^2$ and the moment inertia is

N	2-BRE		3-BRE		4-BRE	
	J_{min}	J_{max}	J_{min}	J_{max}	J_{min}	J_{max}
5	16.0	10.1	36.3	2.9	7.7	0.2
6	52.3	10.5	66.4	3.1	83.0	0.0
7	12.4	11.8	39.1	4.8	33.0	0.5
8	42.7	12.1	63.2	5.0	84.3	1.1
9	9.5	12.3	31.1	6.5	33.7	2.3
10	31.2	11.5	48.6	7.1	65.5	2.6
\mathcal{C}	1.883	3.806	1.560	3.266	0.435	0.000

Table I: Summary of g.s. statistics for minimum and maximum spin, and correlations across different mass numbers N for a single $j = 19/2$ valence space with 2, 3, and 4-body random interactions. The first six rows correspond to $P[J(N)]$ expressed in percent. The lowest row shows correlation \mathcal{C} , for all particle numbers $N = 5, 6, 7, 8, 9$, and 10. Columns reflect the type of ensemble and J_{min} or J_{max} .

particle-number dependent. However, the 3-BRE is expected to be similar to the 2-BRE where the g.s. spin statistics is dictated primarily by the sign of the moment of inertia which equally favors both J_{min} and J_{max} . Although the effect is reduced by incoherent interaction components the preponderance of J_{min} and J_{max} is seen in Fig. 2 and Tab. I. The N -dependence reduces the amount of correlation between systems of different masses, see Tab. I. However, the variations of N which is relatively large and positive are usually insufficient to change the sign in the moment of inertia and therefore correlations between systems of different N are strong.

The angular momentum dependent part in the 4-body forces is given by a more complicated expression with four possible, normally distributed, interaction terms $\langle H^{(2)} \rangle_J = (V_1''N^2 + \tilde{V}_1'N + \tilde{V}_1)J^2 + \tilde{V}_2J^4$. The coefficients $\tilde{V}_2, \tilde{V}_1, \tilde{V}_1'$, and \tilde{V}_1'' are given by correlated normal distributions but because of geometric complexity in the recoupling coefficients and a large number of 4-body interaction parameters these correlations are small. The $P[(J_{max})_0]$ is expected to be significantly reduced, with more chances going to the $P[(J_{min})_0]$ which is always a local minimum for non-negative J^2 , Tab. I and Fig. 1. Only for a large particle number the term $V_1''N^2J^2$ in the Hamiltonian seems to dominate enhancing the $P[(J_{max})_0]$, Tab. I. Because of the strong N -dependence in the moment of inertia and presence of the centrifugal distortion \tilde{V}_2 for $n = 4$ there is much less correlation between g.s. spins for systems with different masses, Tab. I.

The prevailing spin sequence of low-lying states provides yet another information about the coherent symmetry structure of interactions. The chances to find g.s. with spin $J_0 = 0$ followed by the first excited state with $J_1 = 2$ and then by the second excited state $J_2 = 4$ are high. In Tab. II this probability $P[0_0, 2_1, 4_2]$ is shown along with the average ratio of the excitation energies for the 2_1 and 4_2 states. The typical numbers between 2 and 10 % reflect extremely high probability compared to the

	2-BRE		3-BRE		4-BRE	
	P	E_1/E_2	P	E_1/E_2	P	E_1/E_2
6	3.7(2)	0.55	4.2(2)*	0.69	4.4(7)*	0.69
8	4.2(2)*	0.59	5.5(2)	0.67	7.4(11)	0.75
10	2.1(1)*	0.72	5.2(2)	0.69	7.3(10)	0.62

Table II: Probability of finding the three lowest states as a sequence 0,2,4, $P[0_0, 2_1, 4_2]$, labeled as P , expressed in percent, and the ratio of excitation energies between 2_1 and 4_2 states. In all cases the sequence 0,2,4 in the most likely g.s. sequence except for those marked with *.

chances of finding this sequence in a random list of spins. However, the term “sequence” must be used with reservations, the weighted correlational entropy (2) between the joint $P[0_0, 2_1, 4_2]$ and the product of independent $P[0_0]$, $P[2_1]$ and $P[4_2]$ is only about $\mathcal{C}[0_0, 2_1, 4_2] \approx 0.4$ in all cases. The number is comparable to unity showing definite correlations, but they are not strong and the high probability to observe the sequence comes partially from the independently high chances of finding low-lying states with angular momenta 0, 2 and 4. These states do not always form a rotational band which would lead to the ratio of excitation energies $E_1/E_2 = 0.3$. It is likely that the members of the ground state band are often higher in energy and further work is needed to identify them.

We conclude this work by showing the statistics of g.s. quantum numbers for systems with the rotational and isospin symmetries in Fig. 3. The preponderance of the symmetric g.s. with $(JT) = (00)$ is robust for every particle number divisible by 4. Although geometrically (00) state is possible in $j = 19/2$ $N = 10$ it does not appear as g.s. The possibility of α -type correlations is to be investigated in the future. The g.s. in the $N = 9$ system can be explained as a single particle coupled to the $N = 8$ core, thus $T_0 = 1/2$ is favored along with $(9/2\ 9/2)$ and $(49/2\ 1/2)$, the maximum isospin or spin states. For $N = 10$ the $P[(JT)_0]$ is only substantial when either $J = 0$ or $T = 0$ with clear preference to the quantum numbers of two coupled identical $j = 9/2$ fermions ($J + T$ is odd). The preponderance of the minimum or maximum in either spin or isospin appears in all cases. Probability to find g.s. with J_{max} or T_{max} diminish with higher n . The overall picture is in qualitative agreement with the hypothesis discussed above: that the powers of J^2 and T^2 operators appear with random sign as property of the coherent interaction components in each realization. The mixed terms, such as J^2T^2 do not single out a particular (JT) pair. In relation to a more complex shell model studies it is interesting to mention a p -shell ($j = 3/2$ and $j = 1/2$) odd-odd ^{10}B case, where

results are similar to the above study favoring g.s. of coupled two-particle quantum numbers. For a degenerate p levels $P[(1^+0)_0] = 19.1(4)\%$, $P[(3^+0)_0] = 36.6(6)\%$, and $P[(0^+1)_0] = 21.3(5)\%$ for 2-BRE; the same numbers are 25.6(5), 30.5(6), 24.5(5) for 3-BRE; and 30.3(6), 29.9(6), 23.2(5) for 4-BRE, respectively. The chances of $(J^{\pi}T)_0$

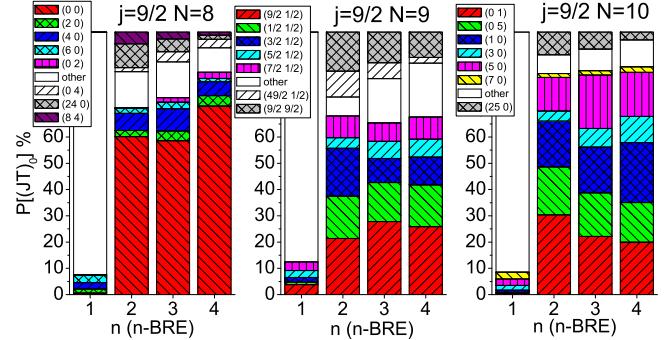


Figure 3: Statistics of g.s. quantum numbers labeled as (JT) for isospin 1/2 fermions on a single $j = 9/2$ level. The panels from left to right correspond to $N = 8, 9$, and 10 . The $n = 1$ is equiprobable distribution. Notations are similar to Fig. 1.

being (1^+0) grow with n , although the statistics has little to do with the realization chosen by nature [1, 2]. Further exploration of more complex models will be reported elsewhere.

To summarize, in this work the ensembles of fermions interacting randomly with symmetry conserving many-body forces have been considered. The preponderance of symmetry dominated ground state was observed. The effect is generally stronger for the higher n -body forces. The possibility of correlated paired ground state is discussed and a surprising enhancement of pairing with three-body forces is observed. An explanation based on the coherent components of interaction is suggested, which qualitatively describes the results. The strong correlations between systems of different particle-number, mass dependence of probabilities, and correlated sequences of states support this theoretical hypothesis. This work opens many future avenues for investigation, studies and ideas [7, 13, 16, 21] from random two-body ensembles can be extended. The onset of coherence: pairing, in isovector or isoscalar form, α -particle four-body clustering, shape properties and vibrations are all interesting and important questions for future investigations.

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